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The Number of Faces of Simplicial Polytopes

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ABSTRACT

The numbers of k-dimensional faces, $f_k \equiv f_k(d)$, $k = -1, 0, ..., d-1$, of a d-dimensional convex polytope satisfy the relations

$$\sum_{i=k}^{d-1} (-1)^i f_i \binom{i+1}{k+1} = (-1)^{d-1} f_k$$

with $f_{-1} = 1$, by convention. These relations are not independent and serve to determine (roughly) half of the f's in terms of the other half. Relations for the f's of upper index in terms of those of lower index found by Branko Grünbaum (in an unpublished report) are rederived here, and recurrences and interrelations of the coefficients in these relations are developed. In addition, the relations for the f's of even index in terms of those of odd index, and vice versa, are found.

1. INTRODUCTION

I have learned from Victor Klee in conversation and in [1] that the numbers of k-dimensional faces $f_k \equiv f_k(d)$, $k = -1, 0, ..., d-1$, of a d-dimensional convex polytope satisfy the relations

$$\sum_{i=k}^{d-1} (-1)^i f_i \binom{i+1}{k+1} = (-1)^{d-1} f_k, \quad k = -1(1)\overline{d-1}, \tag{1}$$

with $f_{-1} = 1$, by convention. These relations are not independent and serve to determine (roughly) half of the f's in terms of the other half. In unpublished work by Branko Grünbaum, the relations determining

f_{n+k}, $k = 0, 1, \ldots, n-1$, $d = 2n$ and $k = 0, 1, \ldots, n$, $d = 2n+1$ in terms of $f_{-1}, f_0, \ldots, f_{n-1}$ have been found. These relations are rederived here, a little more simply, and recurrences and interrelations for their coefficients are developed. In addition, the expressions for the f's of even index in terms of those of odd index, and vice versa, are found.

Grünbaum's result may be rendered as follows:

$$\begin{aligned} f_{n+k} &= \sum_{j=0}^{n-1} (-1)^{n+1+j} f_j A^e_{kj}(n), \quad d = 2n, \quad k = 0, 1, \ldots, n-1 \\ &= \sum_{j=-1}^{n-1} (-1)^{n+1+j} f_j A^o_{kj}(n), \quad d = 2n+1, \quad k = 0, 1, \ldots, n \end{aligned} \tag{2}$$

with

$$\begin{aligned} A^e_{kj}(n) &= \sum_{i=0}^{k} (-1)^i \binom{n-1-i}{k-i} \left[\binom{2n-1-j}{n-1-i} - \binom{2n-1-j}{n+1+i} \right] \\ &= \binom{2n-1-j}{n-1-k} \binom{n-1-j+k}{k} \\ &- \sum_{i=0}^{k} (-1)^i \binom{n-1-i}{k-i} \binom{2n-1-j}{n+1+i} \end{aligned} \tag{3}$$

$$\begin{aligned} A^o_{kj}(n) &= \sum_{i=0}^{k} (-1)^i \binom{n-i}{k-i} \left[\binom{2n-j}{n-i} + \binom{2n-j}{n+1+i} \right] \\ &= \binom{2n-j}{n-k} \binom{n-1-j+k}{k} \\ &+ \sum_{i=0}^{k} (-1)^i \binom{n-i}{k-i} \binom{2n-j}{n+1+i}. \end{aligned} \tag{4}$$

The coefficients $A^e_{kj}(n)$ and $A^o_{kj}(n)$ satisfy the following recurrence

$$A_{kj}(n) - A_{k,j+1}(n) = A_{k,j-1}(n-1) + A_{k-1,j-1}(n-1), \quad k < n, \tag{5}$$

and also, for the full range of Eqs. (2):

$$\begin{aligned} A^e_{kj}(n) &= \binom{n}{k+1} A^e_{0j}(n-1) - A^e_{k+1,j}(n-1), \\ A^o_{kj}(n) &= \left[\binom{n+1}{k+1} + \binom{n}{k} \right] A^o_{0j}(n-1) - A^o_{k+1,j}(n-1). \end{aligned} \tag{6}$$

The even-odd interrelations are

$$f_{d-2-2k} = \sum_{j=0}^{k} A_{k-j} \binom{d-1-2j}{2k+1-2j} f_{d-2j-1}, \tag{7}$$

$$(d-2k) f_{d-1-2k} = \sum_{j=0}^{k} B_{k-j} \binom{d-1-2j}{2k-2j} f_{d-2j-2}, \tag{8}$$

where $A_k = a_{2k}$, with a_k determined by the even ($a_{2k+1} = 0$) exponential generating function

$$\exp xa = 4(e^x + e^{-x} + 2)^{-1}, \quad a^n \equiv a_n,$$

and $B_k = 2b_{2k}$, with b_k a Bernoulli number in the even suffix notation; that is, b_k is determined by

$$\exp xb = x(e^x - 1)^{-1}, \quad b^n \equiv b_n.$$

2. Upper Index Determined Set

It follows from Eq. (1) that

$$\sum_{k=-1}^{r-1} (-1)^{d+k} \binom{d-k-1}{d-r} f_k = \sum_{k=-1}^{r-1} \sum_{i=k}^{d-1} (-1)^{i+k+1} \binom{i+1}{k+1} \binom{d-k-1}{d-r} f_i$$
$$= \sum_{i=-1}^{d-1-r} (-1)^i \binom{d-1-i}{r} f_i. \tag{9}$$

According to Grünbaum, these relations appear in a 1927 paper by D. M. Y. Sommerville [2]. The identity required for their proof is

$$\sum_{k=-1}^{r-1} (-1)^{k+1} \binom{d-k-1}{d-r} \binom{i+1}{k+1}$$
$$= \sum_{k=0}^{r} (-1)^k \binom{d-k}{r-k} \binom{i+1}{k} = \binom{d-1-i}{r},$$

which is one of the variants of the Vandermonde convolution.

The transformation (9) produces identities by pairs. Thus for $d = 3$, the result is

$$\begin{aligned}
-f_{-1} + f_0 - f_1 + f_2 &= f_{-1} \\
-3f_{-1} + 2f_0 - f_1 &= 3f_{-1} - f_0 \\
-3f_{-1} + f_0 &= 3f_{-1} - 2f_0 + f_1 \\
-f_{-1} &= f_{-1} - f_0 + f_1 - f_2.
\end{aligned}$$

The last is the same as the first, the third the same as the second. For $d = 4$:

$$\begin{aligned}
&-f_{-1} + f_0 - f_1 + f_2 - f_3 &&= -f_{-1}\\
&-4f_1 + 3f_0 - 2f_1 + f_2 &&= -4f_{-1} + f_0\\
&-6f_{-1} + 3f_0 - f_1 &&= -6f_{-1} + 3f_0 - f_1\\
&-4f_{-1} + f_0 &&= -4f_{-1} + 3f_0 - 2f_1 + f_2\\
&-f_{-1} &&= -f_{-1} + f_0 - f_1 + f_2 - f_3.
\end{aligned}$$

Again the last is identical with the first, the next last with the second, and the middle is vacuous. In either case, two independent equations are left.

Note that for d even, the terms in f_{-1} cancel, while for d odd they do not. Hence it is convenient to consider the two parities separately.

For $d = 2n$, notice first that the n-th equation, the instance $r = n - 1$ of (9), is

$$\sum_{k=-1}^{n-2} (-1)^k \binom{2n-1-k}{n+1} f_k = \sum_{k=-1}^{n} (-1)^k \binom{2n-1-k}{n-1} f_k$$

or

$$\begin{aligned}
f_n &= nf_{n-1} + \sum_{k=-1}^{n-2} (-1)^{n+k} \left[\binom{2n-1-k}{n+1} - \binom{2n-1-k}{n-1}\right] f_k\\
&= \sum_{k=0}^{n-1} (-1)^{n+1+k} \frac{k+1}{n+1} \binom{2n-k}{n} f_k \qquad (10)
\end{aligned}$$

since

$$\begin{aligned}
&\binom{2n-1-k}{n-1} - \binom{2n-1-k}{n+1}\\
&\qquad = \binom{2n-1-k}{n-1} + \binom{2n-1-k}{n} - \binom{2n-k}{n+1}\\
&\qquad = \binom{2n-k}{n} - \binom{2n-k}{n+1} = \frac{k+1}{n+1} \binom{2n-k}{n}.
\end{aligned}$$

Next the r-th equation after transposing reads

$$\begin{aligned}
&\sum_{j=0}^{n-1} (-1)^j f_j \left[\binom{2n-1-j}{r-1} - \binom{2n-1-j}{r-2-j}\right]\\
&= \sum_{k=0}^{n-r} (-1)^{n+1+k} \binom{n-1-k}{r-1} f_{n+k} \qquad (11)
\end{aligned}$$

or

$$\sum_{j=0}^{n-1} (-1)^{n+1+j} f_j \left[\binom{2n-1-j}{n-1-r} - \binom{2n-1-j}{n+1-r} \right]$$

$$= \sum_{i=0}^{r} (-1)^i \binom{n-1-i}{n-1-r} f_{n+i}. \tag{11a}$$

Multiplying (11a) by the factor

$$(-1)^r \binom{n-1-r}{k-r},$$

and summing from zero to k, gives the first half of (2) (with $A_{kj}^e(n)$ defined by (3)), since

$$\sum_{r=0}^{k} \sum_{i=0}^{r} (-1)^{i+r} \binom{n-1-i}{n-1-r} \binom{n-1-r}{k-r} f_{n+i}$$

$$= \sum_{i=0}^{k} \binom{n-1-i}{n-1-r} f_{n+i} \sum_{r=i}^{k} (-1)^{i+r} \binom{k-i}{r-i}$$

$$= \sum_{i=0}^{k} \binom{n-1-i}{n-1-k} \delta_{ki} f_{n+i} = f_{n+k},$$

with δ_{ki} the Kronecker delta.

For $d = 2n + 1$, first the instance $r = n$ of (9) gives

$$\sum_{j=-1}^{n-1} (-1)^{j+1} f_j \binom{2n-j}{n+1} = \sum_{j=-1}^{n} (-1)^j f_j \binom{2n-j}{n}$$

or

$$f_n = \sum_{j=-1}^{n-1} (-1)^{n+1+j} f_j \left[\binom{2n-j}{n+1} + \binom{2n-j}{n} \right]$$

$$= \sum_{j=-1}^{n-1} (-1)^{n+1+j} f_j \binom{2n+1-j}{n+1}. \tag{12}$$

Next, replacing r by $n - r$ in (9) and transposing leads to

$$\sum_{j=-1}^{n-1} (-1)^{n+1+j} f_j \left[\binom{2n-j}{n-r} + \binom{2n-j}{n+1+r} \right] = \sum_{i=0}^{r} (-1)^i f_{n+i} \binom{n-i}{n-r} \tag{13}$$

and this, multiplied by

$$(-1)^r \binom{n-r}{k-r}$$

and summed from zero to k, gives the second half of (2).

The partial evaluation of the right-hand side of (3) is by

$$\begin{aligned}\sum_{i=0}^{k} (-1)^i \binom{n-1-i}{k-i} \binom{2n-1-j}{n-1-i} &= \binom{2n-1-j}{n-1-k} \sum_{0}^{k} (-1)^i \binom{n+k-j}{k-1} \\ &= \binom{2n-1-j}{n-1-k} \binom{n-1-j+k}{k}.\end{aligned}$$

The partial evaluation of (4) is made in the same way. The recurrence (5) is a consequence of the simplest recurrence for binomial coefficients.

If the second forms of (3) and (4) are rewritten as

$$\begin{aligned}A_{kj}^e(n) &= B_{kj}^e(n) - C_{kj}^e(n),\\ A_{kj}^o(n) &= B_{kj}^o(n) + C_{kj}^o(n),\end{aligned}$$

then, by the simplest binomial recurrence,

$$C_{kj}^o(n) = C_{k,j-1}^e(n) + C_{k-1,j-1}^e(n), \qquad k < n, \tag{14}$$

which implies

$$A_{kj}^o(n) = B_{kj}^e(n+1) - A_{k,j-1}^e(n) - A_{k-1,j-1}^e(n), \qquad k < n. \tag{15}$$

Note that

$$B_{kj}^o(n) = B_{k,j+1}^e(n+1) = \binom{2n-j}{n-k} \binom{n-1-j+k}{k}$$

and that all the B's and C's satisfy recurrence (5).

Tables 1 and 2 show the coefficients $A_{kj}^e(n)$ and $A_{kj}^o(n)$, respectively, the former for $n = 2(1)5$ and the latter for $n = 1(1)4$.

From Table 1 it may be noted (and readily verified) that

$$A_{n-1,j}^e(n) = A_{0j}^e(n-1),$$

since

$$A_{0,n-1}^e(n-1) = 1.$$

Also the last but one of Eqs. (1) implies

$$A_{n-2,j}^e(n) = nA_{n-1,j}^e(n) = nA_{0j}^e(n-1);$$

TABLE 1

THE COEFFICIENTS $A^e_{kj}(n)$

$n = 2$	k/j	0	1			
	0	2	2			
	1	1	1			
$n = 3$	k/j	0	1	2		
	0	5	5	3		
	1	6	6	3		
	2	2	2	1		
$n = 4$	k/j	0	1	2	3	
	0	14	14	9	4	
	1	28	28	17	6	
	2	20	20	12	4	
	3	5	5	3	1	
$n = 5$	h/j	0	1	2	3	4
	0	42	42	28	14	5
	1	120	120	78	36	10
	2	135	135	87	39	10
	3	70	70	75	20	5
	4	14	14	9	4	1

moreover

$$A^e_{n-3,j}(n) = \binom{n}{2} A^e_{0j}(n-1) - A^e_{n-2,j}(n-1),$$

which suggests the relation, the first of Eqs. (6) in the introduction,

$$A^e_{kj}(n) = \binom{n}{k+1} A^e_{0j}(n-1) - A^e_{k+1,j}(n-1). \tag{16}$$

This is proved as follows. First write

$$a_i(n, j) = \binom{2n-3-j}{n-2-i} - \binom{2n-3-j}{n+i}$$

so that

TABLE 2

THE COEFFICIENTS $A^{o}_{kj}(n)$

$n = 1$	k/j	-1	0			
	0	6	3			
	1	4	2			
$n = 2$	k/j	-1	0	1		
	0	20	10	4		
	1	30	15	5		
	2	12	6	2		
$n = 3$	k/j	-1	0	1	2	
	0	70	35	15	5	
	1	168	84	34	9	
	2	140	70	28	7	
	3	40	20	8	2	
$n = 4$	k/j	-1	0	1	2	3
	0	252	126	56	21	6
	1	840	420	182	63	14
	2	1800	540	232	78	16
	3	630	315	135	45	9
	4	140	70	30	10	2

$$A^{e}_{k+1,j}(n-1) = \sum_{i=0}^{k+1} (-1)^i a_i(n, j) \binom{n-2-i}{k+1-i}.$$

Then notice that, abbreviating $a_i(n, j)$ to a_i,

$$\binom{2n-1-j}{n-1-i} - \binom{2n-1-j}{n+1-i} = a_{i-1} + 2a_i + a_{i+1}\,,$$

while

$$\binom{2n-1-j}{n-1} - \binom{2n-1-j}{n+1} = 2a_0 + a_1\,,$$

which is consistent with $a_{-1} = 0$.

Hence

$$\begin{aligned}
A^e_{kj}(n) + A^e_{k+1,j}(n-1) &= \sum_{i=0}^{k} (-1)^i \binom{n-1-i}{k-i} (a_{i-1} + 2a_i + a_{i+1}) \\
&+ \sum_{i=0}^{k+1} (-1)^i \binom{n-2-i}{k+1-i} a_i \\
&= \left[2\binom{n-1}{k} - \binom{n-2}{k-1} + \binom{n-2}{k+1}\right] a_0 \\
&+ \sum_{i=1}^{k} (-1)^i a_i \left[-\binom{n-i}{k+1-i} + 2\binom{n-1-i}{k-i}\right. \\
&- \left.\binom{n-2-i}{k-1-i} + \binom{n-2-i}{k+1-i}\right] \\
&= \binom{n}{k+1} a_0 = \binom{n}{k+1} A^e_{0j}(n-1).
\end{aligned}$$

The proof of the second of Eqs. (6) is closely similar. Note that

$$A^e_{k,n-1}(n) = \binom{n}{k+1}, \quad A^o_{k,n-1}(n) = \binom{n+1}{k+1} + \binom{n}{k};$$

hence (6) may be given the general form

$$A_{kj}(n) = A_{k,n-1}(n) A_{0j}(n-1) - A_{k+1,j}(n-1). \tag{6a}$$

By iterations, Eq. (6a) becomes

$$A_{kj}(n) = \sum_{i=0}^{n-1} (-1)^i A_{k+i,n-i-1}(n-i) A_{0j}(n-1-i), \tag{17}$$

that is

$$\begin{aligned}
A^e_{kj}(n) &= \sum_{i=0}^{n-1} (-1)^i \binom{n-i}{k+i+1} A^e_{0j}(n-1-i), \\
A^o_{kj}(n) &= \sum_{i=0}^{n-1} (-1)^i \left[\binom{n-i+1}{k+i+1} + \binom{n-i}{k+i}\right] A^o_{0j}(n-1-i).
\end{aligned} \tag{18}$$

It is worth noting that $A^e_{0j}(n) = a_{n,n-1}$, where a_{nm} is the simplest ballot number (the number of two-candidate election returns with final vote n for A, m for B, $n \geq m$, such that A is never behind throughout the count).

Finally, an alternate form for the sum $C^e_{kj}(n)$ is as follows

$$
\begin{aligned}
C^e_{kj}(n) &= \sum_{i=0}^{k} (-1)^i \binom{n-1-i}{k-i} \binom{2n-1-j}{n+1+i} \\
&= \sum_{0}^{k} (-1)^i \binom{2n-n-1-i}{n-1-k} \binom{2n-1-j}{n+1+i} \\
&= \sum_{0}^{k} (-1)^i \binom{2n-1-j}{n+1+i} \\
&\quad \times \sum_{s=0}^{n-1-k} (-1)^i \binom{n+1+i}{s} \binom{2n-s}{n-1-k-s} \\
&= \sum_{s=0}^{n-1-k} (-1)^s \binom{2n-1-j}{s} \binom{2n-s}{n-1-k-s} \\
&\quad \times \sum_{i=0}^{k} (-1)^i \binom{2n-1-j-s}{n+1+i-s} \\
&= \sum_{s=0}^{n-1-k} (-1)^s \binom{2n-1-j}{s} \binom{2n-s}{n-1-k-s} \\
&\quad \times \left[\binom{2n-2-j-s}{n-s} + (-1)^k \binom{2n-2-j-s}{n+1+k-s} \right]. \qquad (19)
\end{aligned}
$$

The corresponding result for $C^o_{kj}(n)$ is

$$
\begin{aligned}
C^o_{kj}(n) &= \sum_{j=0}^{n-k} (-1)^s \binom{2n-j}{s} \binom{2n+1-s}{n-k-s} \\
&\quad \left[\binom{2n-1-j-s}{n-s} + (-1)^k \binom{2n-1-j-s}{n+1+k-s} \right]. \qquad (20)
\end{aligned}
$$

3. Even-Odd Relations

Consider first $d = 4$. The set of Eq. (1) is

$$
\begin{aligned}
-f_{-1} + f_0 - f_1 + f_2 - f_3 &= -f_{-1} \\
f_0 - 2f_1 + 3f_2 - 4f_3 &= -f_0 \\
-f_1 + 3f_2 - 6f_3 &= -f_1 \\
f_2 - 4f_3 &= -f_2 \\
-f_3 &= -f_3
\end{aligned}
$$

or

$$\begin{aligned} f_0 - f_1 + f_2 - f_3 &= 0 \\ 2f_0 - 2f_1 + 3f_2 - 4f_3 &= 0 \\ 3f_2 - 6f_3 &= 0 \\ 2f_2 - 4f_3 &= 0. \end{aligned}$$

The last two equations are identical, and imply the identity of the first two; that is, the system may be telescoped to

$$\begin{aligned} f_0 &= f_1 - f_3 \\ f_2 &= 2f_3. \end{aligned}$$

For the general case, the reduced system of equations (reading by twos from the bottom up) may be taken as

$$2f_{d-2} = df_{d-1}, \tag{21}$$

$$2f_{d-4} - \binom{d-1}{2} f_{d-2} = \binom{d}{3} f_{d-1} + (d-2) f_{d-3},$$

where in general

$$\begin{aligned} 2f_{d-2k} &- \sum_{1}^{k-1} \binom{d+1-2k+2j}{2j} f_{d-2k+2j} \\ &= \sum_{j=0}^{k} \binom{d-2j}{2k-1-2j} f_{d-1-2j}, \quad k = 1(1)[d/2]. \end{aligned}$$

To eliminate on the left, notice first that f_{d-2} is eliminated from the first two equations if the first is multiplied by $A_1\binom{d-1}{2}$, the two are added, and $1 + 2A_1 = 0$. For three equations the multipliers in order are $A_2\binom{d-1}{4}$, $A_1\binom{d-3}{2}$, and 1, and the condition on A_2 is

$$1 + \binom{4}{2} A_1 + 2A_2 = 0.$$

These coefficient conditions are both instances of

$$1 + \binom{2k}{2} A_1 + \binom{2k}{4} A_2 + \cdots + \binom{2k}{2k-2} A_{k-1} + 2A_k = 0, \qquad k = 1, 2, \ldots . \tag{22}$$

Setting $A_0 = 1$ and introducing the (even) exponential generating function

$$\exp xa = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \qquad a_{2n} = A_n, \qquad a_{2n+1} = 0,$$

it is found that (22) implies

$$(e^x + e^{-x} + 2) \exp xa = 4, \quad a^n \equiv a_n, \tag{23}$$

or

$$\exp 2xa = \left(\frac{2}{e^x + e^{-x}}\right)^2 = (\exp xE)^2 = \exp x(E + E), \ E^n \equiv E_n, \tag{23a}$$

with E_n the Euler number ($E_{2n+1} = 0$). Eq. (23a) implies

$$2^{2n} A_n = (E + E)^{2n} = \sum_{j=0}^{n} \binom{2n}{2j} E_{2n-2j} E_{2j}. \tag{24}$$

The first few values of the A_n are as follows:

n	0	1	2	3	4	5
A_n	1	$-\frac{1}{2}$	1	$-\frac{17}{4}$	31	$-\frac{691}{2}$

The immediate effect of the elimination is to produce

$$2f_{d-2k-2} = \sum_{j=0}^{k} \binom{d-2j}{2k+1-2j} f_{d-1-2j} \sum_{i=0}^{k-1} \binom{2k+1-2j}{2i} A_i.$$

The inner sum may be simplified as follows. Multiply (23) by e^x and differentiate; then

$$\begin{aligned}(e^x + 1)^2 a \exp xa &= (e^{2x} + 2e^x + 1 - 2e^{2x} - 2e^x) \exp xa \\ &= (1 - e^x)(1 + e^x) \exp xa, \quad a^n \equiv a_n,\end{aligned}$$

or

$$(1 + e^x) a \exp xa = (1 - e^x) \exp xa \tag{25}$$

This implies

$$a_{n+1} + a(a + 1)^n = a_n - (a + 1)^n \tag{26}$$

or

$$\begin{aligned}a_n &= a_{n+1} + a(a + 1)^n + (a + 1)^n = a_{n+1} + (a + 1)^{n+1} \\ &= a_{n+1} + \sum_{0}^{n+1} \binom{n+1}{k} a_k\end{aligned}$$

Since $a_{2n+1} = 0$, it follows that

$$A_n = \sum_{k=0}^{n} \binom{2n+1}{2k} A_k \tag{27}$$

and finally

$$2f_{d-2-2k} = \sum_{j=0}^{k} A_{k-j} \binom{d-2j}{2k+1-2j} f_{d-1-2j}$$

To eliminate on the right of (21) the multipliers are as before with constants C_1, C_2, ... replacing A_1, A_2, ... The equations determining these are:

$$\begin{aligned} 0 &= 1 + \binom{3}{2} C_1 \\ &= 1 + \binom{5}{2} C_1 + \binom{5}{4} C_2 \\ &= 1 + \binom{2k+1}{2} C_1 + \binom{2k+1}{4} C_2 + \cdots + \binom{2k+1}{2k} C_k \end{aligned} \tag{28}$$

Writing $C_n = D_{2n}$ (the notation is chosen to agree with Nörlund) it is found that the exponential generating function for the D's is given by

$$\exp xD = 2x(e^x - e^{-x})^{-1} \tag{29}$$

Note that $\exp xD = \exp - xD$ which implies that all coefficients of odd index vanish. The first few values are:

n	0	1	2	3	4	5
D_{2n}	1	$-\frac{1}{3}$	$\frac{7}{15}$	$-\frac{31}{21}$	$\frac{127}{15}$	$-\frac{2555}{33}$

The coefficients on the left, after elimination, are, in order:

$$2, \quad 1 + 2C_1, \quad 1 + \binom{4}{2} C_1 + 2C_2, \ \ldots;$$

the general term is $D_{2n} + (D + 1)^{2n}$, whose generating function is

$$(e^x + 1) \exp xD = 2x + 2 \exp xb,$$

with $\exp xb = x(e^x - 1)^{-1}$, as in the introduction. Hence

$$B_n = D_{2n} + (D + 1)^{2n} = 2b_{2n}.$$

The first few values of the B_n are as follows:

n	0	1	2	3	4	5
B_n	2	$\frac{1}{3}$	$-\frac{1}{15}$	$\frac{1}{21}$	$-\frac{1}{15}$	$\frac{5}{33}$

References

1. V. Klee, A Combinatorial Analogue of Poincaré's Duality Theorem, *Canad. J. Math.* **16** (1964), 517–531.

2. D. M. Y. Sommerville, The Relations Connecting the Angle-Sums and Volume of a Polytope in Space of *n* Dimensions, *Proc. Roy. Soc. London Ser. A* **115** (1927), 103–119.